Analytic Expression for the Joint x and Q^2 Dependences of the Structure Functions of Deep Inelastic Scattering

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We obtain a good analytic fit to the joint Bjorken-x and Q^2 dependences of ZEUS data on the deep inelastic structure function $F_2(x, Q^2)$. At fixed virtuality Q^2, as we showed previously, our expression is an expansion in powers of $\ln(1/x)$ that satisfies the Froissart bound. Here we show that for each x, the Q^2 dependence of the data is well described by an expansion in powers of $\ln Q^2$. The resulting analytic expression allows us to predict the logarithmic derivatives $(\partial^n F_2^p/(\partial \ln Q^2)^n)_x$ for $n = 1, 2$ and to compare the results successfully with other data. We extrapolate the proton structure function $F_2^p(x, Q^2)$ to the very large Q^2 and the very small x regions that are inaccessible to present day experiments and contrast our expectations with those of conventional global fits of parton distribution functions.

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Introduction. The ability to predict cross sections at very high energies, whether at the CERN Large Hadron Collider or in ultra high energy cosmic ray interactions, depends critically on the reliability of extrapolations from current measurements into regions of much greater virtuality (Q^2) of the elementary scattering processes, and to much smaller values of the fractional longitudinal momentum x carried by the parton constituents of the hadrons. Most high energy predictions are expressed in terms of convolutions of elementary hard-scattering cross sections with parton distribution functions (PDFs). The quantitative reliability of these predictions relies on the x and Q^2 dependences embodied in the universal parton distribution functions extracted from global analyses in perturbative quantum chromodynamics (QCD) of data at lower energies. Important in these global analyses are data from deep-inelastic lepton scattering (DIS) and other reactions, but equally critical are the analytic functional forms assumed for the x dependence of parton distribution functions at small x.

In an earlier paper [1], we analyze the x dependence of the DIS proton structure functions $F_2^p(x, Q^2)$. We begin with the assumption that the x dependence at extremely small x should manifest a $\ln^2(1/x)$ behavior consistent with saturation of the Froissart bound on hadronic total cross sections [2]. Over the ranges of x and Q^2 for which DIS data are available, we show that a very good fit to the x dependence of data from the ZEUS collaboration [3] is obtained for $x \le x_P = 0.09$ and $\frac{Q^2}{x} \gg m^2$ with the expression

$$F_2^p(x, Q^2) = (1-x)\times\left\{\frac{F_P}{1-x_P} + A(Q^2)\ln\left[\frac{x_P}{x}\frac{1-x}{1-x_P}\right] + B(Q^2)\ln^2\left[\frac{x_P}{x}\frac{1-x}{1-x_P}\right]\right\}. \quad (1)$$

The form of Eq. (1) shows explicitly that the x dependence behaves as $\ln^2(1/x)$ at very small x for each value of Q^2. Our successful fits to data at 24 values of Q^2 cover the wide range $0.11 \le Q^2 \le 1200$ GeV^2. The value $x_P = 0.09$ is a scaling point [1] such that the curves for all Q^2 pass through the point $x = x_P$, at which $F_2(x_P, Q^2) = F_P \sim 0.41$. Our logarithmic parameterization of the x dependence at small x implies quite different expectations for high energy hadron cross sections from those based on the inverse power x dependence characteristic of many PDF fits.

In this note, we extend our analysis by making a *joint fit of both* the x and Q^2 dependences of the ZEUS [3] data on $F_2^p(x, Q^2)$. The analytic expression we derive for the x and Q^2 dependences allows us to compute the logarithmic partial derivatives $(\partial^n F_2^p(x, Q^2)/(\partial \ln Q^2)^n)_x$ for $n = 1, 2$. We obtain excellent agreement when comparing our predictions for the first derivative ($n = 1$) with data from the H1 collaboration [4]. We offer predictions for the second derivative ($n = 2$), known as the "curvature". Only 8 parameters—two of which are the scaling value $F_P = 0.41$ at the scaling point $x_P = 0.9$—are needed to fit the joint x and Q^2 dependences. Our expression allows us to extrapolate to very large energies, well beyond the experimental range presently accessible. We obtain cross sections for ultra high energy cosmic ray neutrino reactions that are a factor of 5 smaller than those based on extrapolations of conventional parton distribution functions.

Joint Fit. In Eq. (1) $F_2^p(x, Q^2)$ is written as a sum of terms that are factorizable as functions of Q^2 times functions of x. In this paper we discuss our fit for the functions $A(Q^2)$ and $B(Q^2)$ of Eq. (1).

Our functional form for the x dependence of Eq. (1) is motivated by the $\ln^2(1/x)$ behavior expected by the Froissart bound, as discussed in Ref. [1]. To parametrize the dependence on Q^2 at fixed x, we assume an expansion in powers of $\ln Q^2$, generally consistent with and motivated by the $\ln Q^2$ variation expected in QCD. We note, moreover, that the H1 collaboration [4] determined that, for fixed x, the Q^2 dependence of $F_2^p(x,Q^2)$ is reproduced by the form $F_2^p(x,Q^2) = \alpha_0(x) + \alpha_1(x)\ln(Q^2) + \alpha_2(x)\ln^2(Q^2)$. We therefore expand the functions $A(Q^2)$ and $B(Q^2)$ as

$$\begin{aligned} A(Q^2) &= a_0 + a_1 \ln Q^2 + a_2 \ln^2 Q^2, \\ B(Q^2) &= b_0 + b_1 \ln Q^2 + b_2 \ln^2 Q^2, \end{aligned} \tag{2}$$

terminating these phenomenological expansions at the quadratic level.

We fit simultaneously the x dependence and the Q^2 dependence of the data (Q^2 is expressed in GeV^2 throughout). We determine the 6 real constants a_0, a_1, a_2, b_0, b_1 and b_2 in Eq. (2) using the Sieve algorithm [5], by minimizing the squared Lorentzian,

$$\Lambda_0^2(\boldsymbol{\alpha};\boldsymbol{x}) \equiv \sum_{i=1}^{N} \ln\left\{1 + 0.179\Delta\chi_i^2(x_i;\boldsymbol{\alpha})\right\}, \tag{3}$$

where $\chi^2(\boldsymbol{\alpha};\boldsymbol{x}) \equiv \sum_{i=1}^N \Delta\chi_i^2(x_i;\boldsymbol{\alpha})$, $\Delta\chi_i^2(x_i;\boldsymbol{\alpha}) \equiv \left(\left[\bar{y}_i(x_i;\boldsymbol{\alpha}) - y_i(x_i)\right]/\sigma_i\right)^2$, $\boldsymbol{\alpha}$ is the parameter space vector, and $\bar{y}_i(x_i;\boldsymbol{\alpha})$ is the theoretical value of the measured y_i at x_i, with measurement error σ_i. Using a $\Delta\chi^2_{i\,\max}$ cut of 6, we find (see Table I) a final corrected χ^2/d.f.=1.09, for 169 degrees of freedom (d.f.), a reasonable fit. The data used are 24 ZEUS data sets, with Q^2 =0.11, 0.25, 0.65, 2.7, 3.5, 4.5, 6.5, 8.5, 10, 12, 15, 18, 22, 27, 35, 45, 79, 90, 120, 200, 250, 450, 800 and 1200 GeV^2. The sifting algorithm eliminated 8 outlier points from a total of 183. These 8 points had a total χ^2 of 63.45, illustrating the value of the Sieve algorithm to eliminate outliers.

The quality of our fit to the x and Q^2 dependences of the data for $x \le x_P$, $\frac{Q^2}{x} \gg m^2$ is shown in Fig. 1 where we show a representative plot of 13 of the data sets. Inspection of Fig. 1 shows that the fit is quite good over the large Q^2 range of the ZEUS [3] data. The 6 fit parameters are given in Table I, along with their statistical errors.

Our successful fit provides us with a closed formula for the proton structure function at very small x and very high virtuality—satisfying unitarity at tiny x, since Eq. (1) saturates the Froissart bound.

Evaluation of $\partial F^p_{2\,x}(x,Q^2)/\partial\ln(Q^2)$. Differentiating Eq. (1) with respect to $\ln(Q^2)$, we obtain

$$\begin{aligned} \frac{\partial F^p_{2\,x}(x,Q^2)}{\partial\ln(Q^2)} &= (1-x)\times \\ &\left\{\left(a_1 + 2a_2\ln Q^2\right)\ln\left[\frac{x_P}{x}\frac{1-x}{1-x_P}\right] + \right. \\ &\left.\left(b_1 + 2b_2\ln Q^2\right)\ln^2\left[\frac{x_P}{x}\frac{1-x}{1-x_P}\right]\right\}, \end{aligned} \tag{4}$$

an analytic expression valid for $x \le x_p$ and $Q^2/x \gg m^2$. We show a plot of $\partial F^p_{2\,x}(x,Q^2)/\partial\ln(Q^2)$ in Fig. 2 for a set of values of x, and we compare our expectations to the values measured by the H1 collaboration [4]. We emphasize that the theoretical values have been constrained by ZEUS [3] data alone, and that they are a *prediction* of the H1 results, *not* a fit to these data. We see from Fig. 2 that our predictions based on the ZEUS data are in fine agreement with the normalization and slope of the H1 results.

Curvature. The "curvature" is the second derivative,

$$\frac{\partial^2 F^p_{2\,x}(x,Q^2)}{\partial\ln(Q^2)^2} = (1-x)\times$$

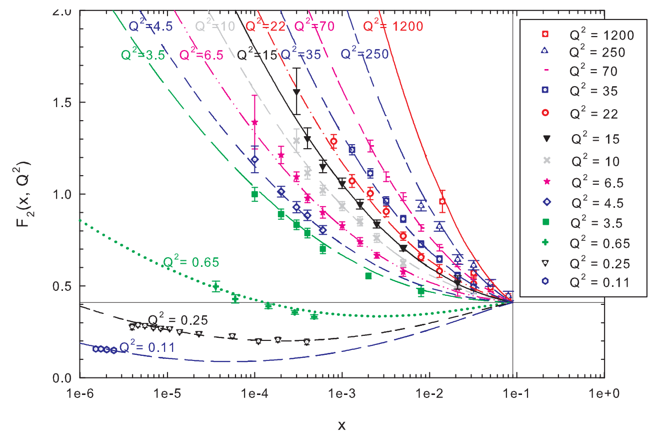

FIG. 1: Fits to the proton structure function data, $F_2^p(x,Q^2)$ vs. x, for 13 values of Q^2. The data are from the ZEUS collaboration [3]. The curves show 13 of our 28 global fits whose parameters are given in Table I. The vertical and horizontal straight lines intersect at the scaling point $x_{\rm P} = 0.09$, $F_2^p(x_{\rm P}) = 0.41$.

TABLE I: Results of a 6-parameter fit to $F_2^p(x,Q^2)$ structure function data[3] using the x and Q^2 behaviors of Eq. (1) and Eq. (2), with Q^2 in GeV^2. The renormalized $\chi^2_{\min}$ per degree of freedom, taking into account the effects of the $\Delta\chi^2_{i\,\max} = 6$ cut [5], is given in the row labeled $\mathcal{R}\times\chi^2_{\min}$/d.f. The errors in the fitted parameters are multiplied by the appropriate $r_{\chi 2}$[5].

Parameters	Values
a_0	$-5.381\times10^{-2}\pm2.17\times10^{-3}$
a_1	$2.034\times10^{-2}\pm1.19\times10^{-3}$
a_2	$4.999\times10^{-4}\pm2.23\times10^{-4}$
b_0	$9.955\times10^{-3}\pm3.09\times10^{-4}$
b_1	$3.810\times10^{-3}\pm1.73\times10^{-4}$
b_2	$9.923\times10^{-4}\pm2.85\times10^{-5}$
$\chi^2_{\min}$	165.99
$\mathcal{R}\times\chi^2_{\min}$	184.2
d.f.	169
$\mathcal{R}\times\chi^2_{\min}$/d.f.	1.09

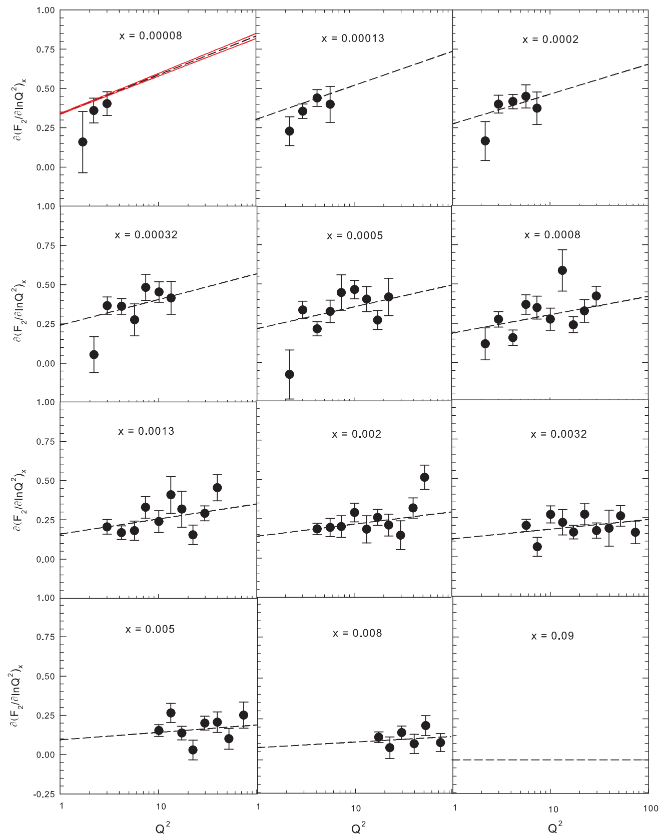

FIG. 2: A plot of the derivative $\partial F^p_{2\,x}(x,Q^2)/\partial \ln(Q^2)$ vs. Q^2, in GeV^2, for selected values of x, compared to data from the H1 collaboration [4]. The exterior lines for $x = 0.00008$ are the error bands associated with the parameter uncertainties of the coefficients of Table I.

$$\left\{2a_2 \ln\left[\frac{x_P}{x}\frac{1-x}{1-x_P}\right] + 2b_2 \ln^2\left[\frac{x_P}{x}\frac{1-x}{1-x_P}\right]\right\}. \quad (5)$$

The form of Eq. (5) indicates that our curvature is independent of Q^2, a consequence of the fact that we truncate the expansions in Eq. (2) at the quadratic level. With our parameterization, the curvature grows like $\ln^2(1/x)$ as x decreases. The signs and magnitudes of a_2 and b_2 determine the sign (positive/negative) of the curvature. In our case, the curvature is positive, and it increases as x decreases, features that are also true in next-to-leading order perturbative QCD [6]. We note, however, that we do not impose or employ QCD evolution in our work. Our predictions are based entirely on our fit to data on $F_2^p(x,Q^2)$. The results of our calculation of curvature are in good agreement with the data shown in Ref. [7]. We remark that curvature is defined somewhat differently in Ref. [7] as the second derivative with respect to $\log_{10}(1+\mathrm{Q}^2/\mathrm{Q}_\mathrm{o}^2)$, instead of with respect to $\ln(Q^2)$.

Extrapolation to Very Small x. In Fig. 3, we present our calculation of $F_2^p(x,Q^2)$ as a function of x for the choices $Q^2 = 25$ GeV^2 and $Q^2 = M_W^2$, where M_W is the mass of the intermediate W boson. The scale $Q = M_W$ is of interest at hadron colliders where it characterizes electroweak processes. It is also the relevant scale in charged-current high energy neutrino interactions [8] where the W boson propagator limits momentum transfers to $Q^2 \sim M_W^2$.

We contrast our expectations with evaluations of $F_2^p(x,Q^2)$ based on the CTEQ6.5 set of parton distribution functions [9]. In our case, the uncertainty bands represent a ± 3 standard deviation variation of our parameters, whereas in the CTEQ6.5 case the bands are obtained from the 40 eigenvector sets that encapsulate the uncertainties of their PDFs.

We observe that the magnitude and x dependence of the CTEQ6.5 and our calculations agree quite well over the range $10^{-3} < x < 0.1$. Both approaches also show the same dependence on Q^2 in this region of x. The agreement is expected since both fit data that are limited to this range of x at large Q^2. The agreement also shows that the logarithmic expansion we use to describe x dependence and the inverse power behavior of the CTEQ form cannot be distinguished numerically over the finite range $10^{-3} < x < 0.1$. However, the two expectations clearly diverge considerably when extrapolated to values of x as low as 10^{-8}.

In the parton model, the decomposition of the structure function $F_2^p(x,Q^2)$ at very small x is dominated by the sea quark $q(x,Q^2)$ and sea antiquark $\bar{q}(x,Q^2)$ densities. Although we do not decompose our F_2^p into parton distributions, this dominance allows us to conclude from Fig. 3 that our sea quark (and antiquark) distributions will be about a factor of 5 smaller than those in CTEQ6.5 at $x \sim 10^{-8}$ and $Q^2 \sim M_W^2$. In ultra high energy charged-current neutrino interactions [8], $x \sim M_W^2/2mE_\nu$, and the cross section on (effectively isoscalar) nucleons, is directly proportional to the sum $\bar{u}(x,M_W^2)+\bar{d}(x,M_W^2)$, where $\bar{u}$ and $\bar{d}$ are the up and down antiquark distributions. Our extrapolation in Fig. 3 shows that the expected cross sections at $x \sim 10^{-8}$ in our case will be about a factor of 5 smaller than those based on CTEQ6.5 [10].

Summary. The Bjorken-x dependence of the DIS proton structure function $F_2^p(x,Q^2)$ measured by the ZEUS collaboration is consistent with a $\ln^2(1/x)$ dependence at small values of x, compatible with saturation of the Froissart bound at each value of Q^2. We parametrize successfully the joint x and Q^2 dependences of F_2 for $x \le x_P \sim 0.09$ and $\frac{Q^2}{x} \gg m^2$, using the compact factorized expression in Eq. (1), with the Q^2 variation expressed in Eq. (2). Our analytic expression has only 6 parameters (plus the scaling point x_P and the value $F_2^p(x_P)$ at the scaling point). We compute the first and second derivatives of $F_2^p(x,Q^2)$ with respect to $\ln Q^2$ at small x. Our predictions of these quantities are in good agreement with the measurements by the H1 collaboration [4]. We extrapolate our expression for $F_2(x,Q^2)$ down to the very

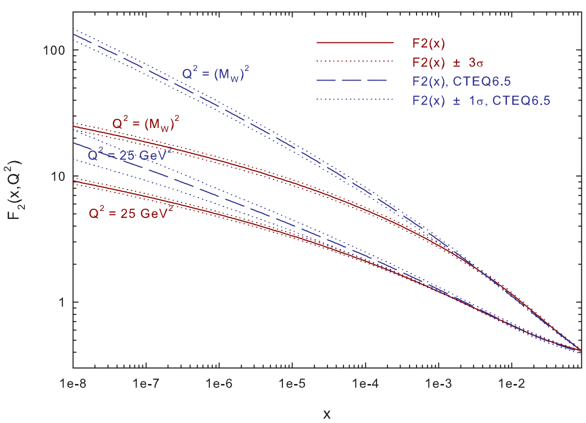

FIG. 3: A plot of $F_2^p(x,Q^2)$ vs. x at $Q^2 = 25$ GeV^2 and $Q^2 = M_W^2$, where M_W is the mass of the W boson, along with results based on the CTEQ6.5M parton distribution functions [9].

small value $x = 10^{-8}$ and compare our expectations to those based on the CTEQ6.5M set of parton distribution functions [9].

Under the assumption that the Froissart bound applies to the virtual photon cross section $\sigma(\gamma^* p)$, a $\ln^2(1/x)$ behavior is as singular as is allowed for the very small x behavior of $F_2^p(x,Q^2)$. However, it is difficult to reconcile a $\ln^2(1/x)$ behavior at very small x for $F_2^p(x,Q^2)$ [and for the gluon distribution $g(x,Q^2)$] at all Q^2 with the Dokshitzer Gribov Lipatov Altarelli Parisi (DGLAP) evolution equation [11] at next-to-leading order in QCD, owing to the singular nature of parton splitting functions at small x. Global analyses of parton distribution functions based on DGLAP evolution, such as CTEQ6.5, begin with the assumption of an inverse power behavior for the small x dependences of the quark, antiquark, and gluon densities, behavior that is more singular than is allowed by the Froissart bound. The assumed inverse-power behavior leads to the very different expectations shown in Fig. 3, where they are seen to diverge for $x \lesssim 10^{-3}$. To the extent that the $\ln^2(1/x)$ behavior is preferable theoretically, we question the reliability at very small x of parton distribution functions based on an assumed inverse power behavior.

The $\ln^2(1/x)$ behavior that we show is consistent with the DIS data could be a signal for the onset of the physics of saturation at high parton densities. Theoretical effort is warranted to devise a QCD evolution framework compatible with $\ln^2(1/x)$ behavior of parton densities at very small x, and experimental programs should be pursued to measure the x and Q^2 variations of structure functions at much smaller values of x than are currently explored.

Our next goals include a reanalysis of all available data for $F_2^p(x,Q^2)$ and $\partial F^p_{2\,x}(x,Q^2)/\partial \ln(Q^2)$ for $x \le x_P$, in ep, μp, and νp deep inelastic scattering, in order to obtain a joint fit to both the x and Q^2 dependences, constrained by the Froissart bound and the scaling point. This work should allow us to make more accurate predictions of the proton structure function at very small x and very large Q^2, regions beyond the reach of existing accelerators. We also will investigate the domain of compatibility in Bjorken-x of a $\ln^2 1/x$ behavior of $F_2^p(x,Q^2)$ at very small x with quark and gluon distribution functions obtained in the standard fashion, e.g., in Ref. [9], with DGLAP evolution and an assumed inverse power behavior of PDFs.

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